

Modular classes of Lie algebroid morphisms

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Abstract We study the behavior of the modular class of a Lie algebroid under general Lie algebroid morphisms by introducing the relative modular class. We investigate the modular classes of pull-back morphisms and of base-preserving morphisms associated to Lie algebroid extensions. We also define generalized morphisms, including Morita equivalences, that act on the 1-cohomology, and observe that the relative modular class is a coboundary on the category of Lie algebroids and generalized morphisms with values in the 1-cohomology.

Introduction

The modular class of a Poisson manifold first appeared, without a name, in the work of Koszul [19]. The name was proposed in [26], where Koszul's class was shown to be the obstruction to the existence of a smooth Poisson trace, and hence the Poisson analogue of the modular automorphism group of a von Neumann algebra. The extension of the modular class to Lie algebroids was also proposed there.

This extension was studied in detail by Evens, Lu and Weinstein [7], who showed that the modular class of the cotangent Lie algebroid of a Poisson manifold is twice that of the Poisson manifold. The theory was then developed in the framework of Lie-Rinehart algebras by Huebschmann [14], and the role of Batalin-Vilkovisky algebras was emphasized in [27] and [15].

The modular class of a base-preserving morphism of Lie algebroids was defined in [12], using an approach similar to that of [15], and studied in [17], where it is called a relative modular class, further explaining results of [16] on the modular classes of twisted Poisson structures on Lie algebroids.

In this paper we extend this relative notion of modular class to Lie algebroid morphisms which are not necessarily base-preserving. We look at many examples and study the properties of this new class. We introduce a category of generalized morphisms of Lie algebroids which includes the weak Morita equivalences of Ginzburg [10] along with ordinary morphisms, and we investigate the modular class in the context of this category.

In Section 1, we recall the definitions of Lie algebroid morphisms and representations, and we prove that representations can be pulled back (Proposition 1.3). We then recall the definition of the characteristic class of a Lie

algebroid representation on a line bundle and that of the modular class of a Lie algebroid, and we study the behavior of the characteristic class of a representation under pull-back (Proposition 2.1). We can then define the modular class of a Lie algebroid morphism (Definition 2.2), an extension of the definition for base-preserving morphisms. In Section 2.3, the modular class of a Lie algebroid morphism is shown to be the characteristic class of a Lie algebroid representation which we construct explicitly (Theorem 2.3).

Section 3 is devoted to the modular classes of pull-back Lie algebroid morphisms; these are the ones for which, in some sense, only the base manifold changes. We recall the notion of pull-back of a Lie algebroid (Definition 3.2) following [22]. We first prove that the modular class of a pull-back morphism vanishes when the morphism covers a submersion (Proposition 3.8), then we prove (Theorem 3.10) that this result is still valid in the more general case of the transverse maps defined in Definition 3.5. A counter-example (Section 3.3) shows that the result is not true in general when the base map fails to be transverse. We also show that a surjective submersion induces an isomorphism from the 1-cohomology of a Lie algebroid to that of its pull-back (Proposition 3.15).

In Section 4, we consider some base-preserving Lie algebroid morphisms, and we show that the modular class of a Lie algebroid extension of a transitive Lie algebroid B by a unimodular Lie algebroid C can be computed in terms of a representation of B on a line bundle (Theorem 4.5), generalizing a theorem in [18].

In Section 5, we define generalized Lie algebroid morphisms and the Morita equivalence of Lie algebroids, as well as the modular class of a generalized morphism with connected and simply-connected fibers. We prove that the modular classes of isomorphic generalized morphisms are equal, and give a formula for the modular class of the composition of two generalized morphisms (Theorem 5.4). It follows from the definition and from Proposition 3.8 that the modular class of a Morita equivalence with connected and simply-connected fibers vanishes.

In the Appendix (Section 6), we describe the modular class of morphisms of Lie algebroids as a 1-coboundary on the category of Lie algebroids.

There are many interesting questions which remain concerning the modular classes of Lie algebroids and their morphisms.

1. The consequences of the unimodularity of a Lie algebroid for the existence of a trace on an algebra quantizing the Poisson algebra of the dual vector bundle have been studied in [24]. What is the meaning of the modular class of a morphism in terms of these algebras?
2. Modular classes have been shown ([7] [14] [27]) to be closely related to duality between Lie algebroid homology and cohomology, and the consequences of the unimodularity of a Poisson variety for the Van den Bergh dualizing module of its deformation quantization have been

studied in [5]. What is the meaning of the modular class of a morphism, in this context?

3. The modular class is the first in a sequence of higher cohomology classes of Lie algebroids which appear in the work of Kubarski (see [20]), Fernandes [8] and Crainic [4]. How do these higher classes behave under morphisms?

Conventions. All the manifolds, bundles and maps will be assumed to be smooth. We denote the space of (smooth global) sections of a real vector bundle E by ΓE , and its dual vector bundle by E^* .

A Lie algebroid is denoted by a triple $(A \rightarrow M, \rho_A, [\ , \]_A)$, where $A \rightarrow M$ is a vector bundle, ρ_A is the anchor map, and $[\ , \]_A$ is the Lie bracket of sections. When no confusion is possible, we shall abbreviate $[\ , \]_A$ to $[\ , \]$. We often denote a Lie algebroid simply by $A \rightarrow M$ or A , in which case the anchor will be understood to be ρ_A .

We shall call a topological space 1-connected if it is both connected and simply-connected.

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1 Morphisms and representations of Lie algebroids

Base-preserving morphisms between Lie algebroids are simply vector bundle maps inducing homomorphisms between Lie algebras of sections. More general morphisms, covering maps between manifolds, were defined by Higgins and Mackenzie [13]. Vaintrob [25] formulated an equivalent condition in terms of vector fields on supermanifolds. A third formulation, close to Vaintrob's, was given by Chen and Liu [3], who proved the equivalence of their definition to that of Higgins and Mackenzie.

Modular classes of base-preserving morphisms of Lie algebroids were defined and studied in [12] and [17]. Here we review the definition of general Lie algebroid morphisms, and we extend the notion of modular class to them.

1.1 Morphisms of Lie algebroids

Let $A \rightarrow M$ and $B \rightarrow N$ be real vector bundles, and let (Φ, ϕ) be a vector bundle map,

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi} & N \end{array}$$

Such a map induces a pull-back operator $\tilde{\Phi}^* : \Gamma(B^*) \rightarrow \Gamma(A^*)$ on sections, defined by

$$\langle (\tilde{\Phi}^* \beta)_m, a_m \rangle = \langle (\beta \circ \phi)(m), \Phi(a_m) \rangle, \quad (1.1)$$

for $\beta \in \Gamma(B^*)$, $m \in M$ and $a \in \Gamma A$. This operator is induced by the base-preserving morphism of vector bundles over M , $\Phi^* : \phi^! B^* \rightarrow A^*$, where $\phi^! B^*$ is the pull-back of B^* under ϕ . We may extend $\tilde{\Phi}^*$ to an exterior algebra homomorphism, $\wedge^\bullet \tilde{\Phi}^* : \Gamma(\wedge^\bullet B^*) \rightarrow \Gamma(\wedge^\bullet A^*)$, where we set $(\wedge^0 \tilde{\Phi}^*)(f)$ to be $f \circ \phi$, for $f \in C^\infty(M)$. Recall that, for any Lie algebroid $(A \rightarrow M, \rho_A, [\cdot, \cdot]_A)$, the Lie algebroid differential d_A is defined by

$$\begin{aligned} (d_A \alpha)(a_0, a_1, \dots, a_k) &= \sum_{i=0}^k (-1)^i \rho_A(a_i) \cdot \alpha(a_0, \dots, \hat{a}_i, \dots, a_k) \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([a_i, a_j]_A, a_0, \dots, \hat{a}_i, \dots, \hat{a}_j, \dots, a_k), \end{aligned}$$

for $\alpha \in \Gamma(\wedge^k A^*)$, $k \in \mathbb{N}$, $a_0, \dots, a_k \in \Gamma A$. The differential d_A turns $\Gamma(\wedge^\bullet A^*)$ into a complex, whose cohomology is called the *Lie algebroid cohomology* and is denoted by $H^\bullet(A)$. It is natural to make the following definition.

Definition 1.1. *Let $A \rightarrow M$ and $B \rightarrow N$ be Lie algebroids, and let (Φ, ϕ) be a vector bundle map from A to B . The pair (Φ, ϕ) is a morphism of Lie algebroids if the map $\wedge^\bullet \tilde{\Phi}^* : (\Gamma(\wedge^\bullet B^*), d_B) \rightarrow (\Gamma(\wedge^\bullet A^*), d_A)$ is a chain map.*

When (Φ, ϕ) is base-preserving, *i.e.*, ϕ is an identity map, $\tilde{\Phi}^*$ is the usual dual of Φ , and the condition that $\wedge^\bullet \tilde{\Phi}^*$ be a chain map is equivalent to the usual definition of a morphism of Lie algebroids as a vector bundle map preserving anchors and brackets [21] [22]. In fact, Definition 1.1 is just the “ordinary language” version of the definition due to Vaintrob [25]: (Φ, ϕ) is a morphism if the homological vector fields d_A and d_B are $\Pi\Phi$ -related, where $\Pi\Phi$ is Φ considered as a morphism from ΠA to ΠB , the supermanifolds obtained from A and B by “making the fibers odd,” *i.e.*, where the functions on ΠA are the sections of $\wedge^\bullet A^*$, and the functions on ΠB are the sections of $\wedge^\bullet B^*$.

To summarize this section, we may say that the morphisms make Lie algebroids over all manifolds a category \mathcal{Alg} . The Lie algebroid complex is a contravariant functor from this category to the category of complexes of vector spaces over \mathbb{R} , and Lie algebroid cohomology is a contravariant functor from \mathcal{Alg} to the category of graded real vector spaces. We discuss this categorical approach further in the Appendix.

1.2 Representations of Lie algebroids

It is useful to think of \mathcal{Alg} as an enlargement of the category of manifolds and smooth maps, which form a full subcategory when we identify each manifold M with its tangent bundle Lie algebroid. The restriction of Lie algebroid cohomology is, of course, just de Rham cohomology. Other

full subcategories are that of Lie algebras, for which the Lie algebroid cohomology is the Chevalley-Eilenberg cohomology, and that of the zero Lie algebroids.

The theory of flat connections on vector bundles over manifolds becomes a representation theory when we pass from manifolds to Lie algebroids, and this representation theory generalizes that of Lie algebras.

We recall that a *representation* [21] [22] of a Lie algebroid A on a vector bundle E , both over the base M , is an \mathbb{R} -bilinear map $D : \Gamma A \times \Gamma E \rightarrow \Gamma E$, denoted $(a, x) \mapsto D(a)x$, or simply $a \cdot x$ when no confusion is possible, such that, for all $f \in C^\infty(M)$, $a, a_1, a_2 \in \Gamma A$ and $x \in \Gamma E$,

$$\begin{cases} (i) & (fa) \cdot x = f(a \cdot x) , \\ (ii) & a \cdot (fx) = f(a \cdot x) + (\rho_A a)(f) x , \\ (iii) & a_1 \cdot (a_2 \cdot x) - a_2 \cdot (a_1 \cdot x) = [a_1, a_2]_A \cdot x . \end{cases}$$

Equivalently, a *representation of A on E* , also called a *flat A -connection on E* or an *A -module structure on E* , is a Lie algebroid morphism over M from A to $\mathcal{D}E$, the Lie algebroid of derivations on E whose sections are the covariant differential operators (CDO's) or derivative endomorphisms of ΓE (see [22]). More generally, an *A -connection on E* is a vector bundle morphism over M from A to $\mathcal{D}E$, *i.e.*, a map satisfying (i) and (ii) above. We also recall the following constructions.

- If D is a representation of A on E , then the dual representation D^* is the representation of A on E^* defined by

$$\langle D^*(a)\xi, x \rangle = - \langle \xi, D(a)x \rangle + (\rho_A a)(\langle \xi, x \rangle) ,$$

for all $a \in \Gamma A$, $x \in \Gamma E$, $\xi \in \Gamma(E^*)$.

- If D_1 and D_2 are representations of A on vector bundles over M , E_1 and E_2 , then $D_1 \otimes D_2$ is the representation of A on $E_1 \otimes E_2$ defined by

$$(D_1 \otimes D_2)(a)(e_1 \otimes e_2) = D_1(a)(e_1) \otimes e_2 + e_1 \otimes D_2(a)(e_2) ,$$

for all $a \in \Gamma A$, $e_1 \in \Gamma(E_1)$ and $e_2 \in \Gamma(E_2)$.

We call an \mathbb{R} -linear endomorphism of $\Gamma(\wedge^\bullet A^* \otimes E)$ of degree 1 and of square zero a *differential* on $\Gamma(\wedge^\bullet A^* \otimes E)$. The following generalization of a well-known characterization of flat connections proves that the definition of a representation of a Lie algebroid A adopted above is equivalent to that of an A -module in [25].

Proposition 1.2. *Any differential $d_{A,E}$ on $\Gamma(\wedge^\bullet A^* \otimes E)$ satisfying*

$$d_{A,E}(\alpha \otimes x) = d_A \alpha \otimes x + (-1)^{|\alpha|} \alpha \otimes d_{A,E} x , \quad (1.2)$$

for all $\alpha \in \Gamma(\wedge^\bullet A^*)$ of degree $|\alpha|$, and $x \in \Gamma E$, gives rise to a representation of A on E defined by

$$a \cdot x = \iota_a(d_{A,E} x) , \quad (1.3)$$

for $a \in \Gamma A$ and $x \in \Gamma E$, where ι denotes the interior product. Conversely, each representation of A on E gives rise in this way to a differential on $\Gamma(\wedge^\bullet A^* \otimes E)$ satisfying (1.2).

Proof. The \mathbb{R} -linear map $d_{A,E}|_{\Gamma E} : \Gamma E \rightarrow \Gamma(A^* \otimes E)$ satisfies, for each $x \in \Gamma E$ and $f \in C^\infty(M)$,

$$d_{A,E}(fx) = fd_{A,E}x + d_A f \otimes x , \quad (1.4)$$

and therefore $(a, x) \mapsto a \cdot x$ satisfies (i) and (ii) above.

Conversely, given a representation, the map $d_{A,E}$ is well-defined on ΓE because of (i) and satisfies (1.4) because of (ii). It can then be uniquely extended to an \mathbb{R} -linear endomorphism of degree 1 of $\Gamma(\wedge^\bullet A^* \otimes E)$ satisfying (1.2).

We claim that $(d_{A,E})^2 = 0$ is equivalent to the flatness property (iii). In fact, expressing $d_{A,E} x$, for $x \in \Gamma E$, locally as a finite sum $d_{A,E} x = \sum_k \alpha_k \otimes y_k$, with $\alpha_k \in \Gamma(A^*)$ and $y_k \in \Gamma E$, yields, for a and $a' \in \Gamma A$,

$$\begin{aligned} \iota_{a'} \iota_a(d_{A,E}(d_{A,E} x)) &= \sum_k \iota_{a'} \iota_a(d_A \alpha_k \otimes y_k - \alpha_k \otimes d_{A,E} y_k) \\ &= \sum_k ((d_A \alpha_k)(a, a') y_k + \langle \alpha_k, a' \rangle \iota_a d_{A,E} y_k - \langle \alpha_k, a \rangle \iota_{a'} d_{A,E} y_k) , \end{aligned}$$

while

$$\begin{aligned} a \cdot (a' \cdot x) - a' \cdot (a \cdot x) &= \sum_k (\iota_a d_A \langle \alpha_k, a' \rangle y_k - \iota_{a'} d_A \langle \alpha_k, a \rangle y_k \\ &\quad + \langle \alpha_k, a' \rangle \iota_a d_{A,E} y_k - \langle \alpha_k, a \rangle \iota_{a'} d_{A,E} y_k) . \end{aligned}$$

Therefore

$$a \cdot (a' \cdot x) - a' \cdot (a \cdot x) - [a, a'] \cdot x = \iota_{a'} \iota_a(d_{A,E}(d_{A,E} x)) ,$$

which proves the claim. \square

This proof shows that, more generally, the A -connections on E are in one-to-one correspondence with the \mathbb{R} -linear endomorphisms of degree 1 of $\Gamma(\wedge^\bullet A^* \otimes E)$ satisfying (1.2).

If $A \rightarrow M$ and $B \rightarrow N$ are Lie algebroids and D_A (resp., D_B) is a representation of $A \rightarrow M$ (resp., $B \rightarrow N$) on a vector bundle $E \rightarrow M$ (resp., $F \rightarrow N$), a vector bundle map $\Psi : E \rightarrow F$ is said to be a *morphism of representations from D_A to D_B covering the Lie algebroid morphism (Φ, ϕ)* from $A \rightarrow M$ to $B \rightarrow N$ if $\Psi : E \rightarrow F$ covers $\phi : M \rightarrow N$, and if the map $\wedge^\bullet \tilde{\Phi}^* \otimes \tilde{\Psi}^*$ is a chain map from the complex $(\Gamma(\wedge^\bullet B^* \otimes F^*), d_{B,F^*})$ to the

complex $(\Gamma(\wedge^\bullet A^* \otimes E^*), d_{A,E^*})$, i.e., if the following diagram commutes.

$$\begin{array}{ccc} \Gamma(\wedge^\bullet B^* \otimes F^*) & \xrightarrow{d_{B,F^*}} & \Gamma(\wedge^{\bullet+1} B^* \otimes F^*) \\ \downarrow \wedge^\bullet \tilde{\Phi}^* \otimes \tilde{\Psi}^* & & \downarrow \wedge^{\bullet+1} \tilde{\Phi}^* \otimes \tilde{\Psi}^* \\ \Gamma(\wedge^\bullet A^* \otimes E^*) & \xrightarrow{d_{A,E^*}} & \Gamma(\wedge^{\bullet+1} A^* \otimes E^*) \end{array} \quad (1.5)$$

If $M = N = \{\text{pt}\}$, the commutativity of (1.5) reduces to the condition that Ψ be a morphism from the A -module E to the B -module F covering the Lie algebra morphism $\Phi : A \rightarrow B$, $\Psi(D_A(a \otimes v)) = D_B(\Phi(a) \otimes \Psi(v))$, for all $a \in A$ and $v \in E$.

1.3 Pull-back of a representation by a Lie algebroid morphism

Let $F \rightarrow N$ be a vector bundle, and let $\phi^! F \rightarrow M$ be its pull-back by a map $\phi : M \rightarrow N$. Any section $y \in \Gamma F$ pulls back to the section $\phi^! y \in \Gamma(\phi^! F)$, where $\phi^! y = y \circ \phi$. The next proposition generalizes the facts that flat bundles pull back to flat bundles and that Lie algebra representations pull back to representations.

Proposition 1.3. *If D is a representation of the Lie algebroid $B \rightarrow N$ on $F \rightarrow N$, and if (Φ, ϕ) is a Lie algebroid morphism from $A \rightarrow M$ to $B \rightarrow N$, then there exists a unique representation $\Phi^! D$ of A on $\phi^! F$ such that, for all $a \in \Gamma A$, $b \in \Gamma B$ satisfying $\Phi \circ a = b \circ \phi$, and $y \in \Gamma F$,*

$$a \cdot (\phi^! y) = \phi^! (b \cdot y) . \quad (1.6)$$

Explicitly, $(\Phi^! D)(a)(\phi^! y) = \phi^! (D(b)y)$.

Proof. We shall define the representation $\Phi^! D$ by means of the associated differential $d_{A,\phi^! F} : \Gamma(\phi^! F) \rightarrow \Gamma(A^* \otimes \phi^! F)$ described in Proposition 1.2. We first define the map $d_{A,\phi^! F}$ on the sections of $\phi^! F$ of the form $\phi^! y$, for $y \in \Gamma F$, by

$$d_{A,\phi^! F}(\phi^! y) = (\tilde{\Phi}^* \otimes \phi^!)(d_{B,F} y) . \quad (1.7)$$

For $f \in C^\infty(M)$, we set

$$d_{A,\phi^! F}(f \otimes \phi^! y) = f \otimes d_{A,\phi^! F}(\phi^! y) + d_A f \otimes \phi^! y , \quad (1.8)$$

A computation that uses the fact that (Φ, ϕ) is a morphism of Lie algebroids, which implies that $\tilde{\Phi}^*(d_B h) = d_A(h \circ \phi)$ for $h \in C^\infty(N)$, shows that

$$d_{A,\phi^! F}((h \circ \phi)f \otimes \phi^! y) = d_{A,\phi^! F}(f \otimes \phi^! (hy)) .$$

Therefore, there is a well-defined linear map $d_{A,\phi^!F}$ to $\Gamma(A^* \otimes \phi^!F)$ from $(C^\infty(M) \otimes_{\mathbb{R}} \Gamma F)/\phi^!(C^\infty(N))$, which is isomorphic to $\Gamma(\phi^!F)$ as a $C^\infty(M)$ -module. Because of (1.8), this map can be extended to an \mathbb{R} -linear endomorphism of degree 1 of $\Gamma(\wedge^\bullet A^* \otimes \phi^!F)$ satisfying (1.2). A straightforward computation shows that this map is of square zero. The corresponding representation satisfies, for $y \in \Gamma F$,

$$a \cdot (\phi^!y) = \iota_a d_{A,\phi^!F}(\phi^!y) = \iota_a(\tilde{\Phi}^* \otimes \phi^!)(d_{B,F} y) .$$

Assuming, without loss of generality, that $d_{B,F} y = \beta \otimes z$, with $\beta \in \Gamma B^*$ and $z \in \Gamma F$, and assuming that $\Phi \circ a = b \circ \phi$, we obtain

$$a \cdot (\phi^!y) = \langle \tilde{\Phi}^* \beta, a \rangle \phi^!z = (\langle \beta, b \rangle \circ \phi) \phi^!z = \phi^!(\langle \beta, b \rangle z) = \phi^!(b \cdot y) ,$$

as required. This property defines the pull-back representation uniquely. \square

See Remark 3.13 below for a simple proof of this proposition in the case where ϕ is a transverse map.

When $d_{B,F}$ and $d_{A,\phi^!F}$ are the differentials associated by Proposition 1.2 to a representation D of $B \rightarrow N$ on F and the pull-back representation $\Phi^!D$ of $A \rightarrow M$ on $\phi^!F$, the following diagram commutes.

$$\begin{array}{ccc} \Gamma(\wedge^\bullet B^* \otimes F) & \xrightarrow{d_{B,F}} & \Gamma(\wedge^{\bullet+1} B^* \otimes F) \\ \downarrow \wedge^\bullet \tilde{\Phi}^* \otimes \phi^! & & \downarrow \wedge^{\bullet+1} \tilde{\Phi}^* \otimes \phi^! \\ \Gamma(\wedge^\bullet A^* \otimes \phi^!F) & \xrightarrow{d_{A,\phi^!F}} & \Gamma(\wedge^{\bullet+1} A^* \otimes \phi^!F) \end{array} \quad (1.9)$$

Corollary 1.4. *The canonical projection $\phi_F: \phi^!F \rightarrow F$ is a morphism of representations from D to $\Phi^!D$ covering the Lie algebroid morphism (Φ, ϕ) .*

Proof. Let $\Phi^!(D^*)$ be the pull-back by (Φ, ϕ) of the representation D^* of $B \rightarrow N$ on F^* , which is a representation of $A \rightarrow M$ on $\phi^!(F^*)$. By (1.9), the following diagram commutes.

$$\begin{array}{ccc} \Gamma(\wedge^\bullet B^* \otimes F^*) & \xrightarrow{d_{B,F^*}} & \Gamma(\wedge^{\bullet+1} B^* \otimes F^*) \\ \downarrow \wedge^\bullet \tilde{\Phi}^* \otimes \phi^! & & \downarrow \wedge^{\bullet+1} \tilde{\Phi}^* \otimes \phi^! \\ \Gamma(\wedge^\bullet A^* \otimes \phi^!(F^*)) & \xrightarrow{d_{A,\phi^!(F^*)}} & \Gamma(\wedge^{\bullet+1} A^* \otimes \phi^!(F^*)) \end{array}$$

When $\phi^!(F^*)$ is identified to $(\phi^!F)^*$ as vector bundles, the map $\phi^!: \Gamma(F^*) \rightarrow \Gamma(\phi^!(F^*))$ is identified with $(\widetilde{\phi_F})^*: \Gamma(F^*) \rightarrow \Gamma((\phi^!F)^*)$, and the previous commutative diagram becomes:

$$\begin{array}{ccc} \Gamma(\wedge^\bullet B^* \otimes F^*) & \xrightarrow{d_{B,F^*}} & \Gamma(\wedge^{\bullet+1} B^* \otimes F^*) \\ \downarrow \wedge^\bullet \tilde{\Phi}^* \otimes (\widetilde{\phi_F})^* & & \downarrow \wedge^{\bullet+1} \tilde{\Phi}^* \otimes (\widetilde{\phi_F})^* \\ \Gamma(\wedge^\bullet A^* \otimes (\phi^!F)^*) & \xrightarrow{d_{A,(\phi^!F)^*}} & \Gamma(\wedge^{\bullet+1} A^* \otimes (\phi^!F)^*) \end{array}$$

By (1.5), the claim is proved. \square

2 Modular classes of morphisms

2.1 Modular classes of Lie algebroids

We recall some results of [7]. When D is a representation of the Lie algebroid $A \rightarrow M$ on an orientable line bundle $L \rightarrow M$, and when λ is a nowhere-vanishing section of L , the section α_λ of A^* defined by

$$\langle \alpha_\lambda, a \rangle \lambda = a \cdot \lambda ,$$

for all $a \in \Gamma A$, is d_A -closed. As above, we have denoted $D(a)\lambda$ by $a \cdot \lambda$. We call α_λ the *characteristic cocycle* associated to the representation D and the section λ . Its class in the 1-cohomology of A is independent of the choice of λ . We call it the *characteristic class* of the representation D and denote it by $\text{char } D$. When L is not orientable, the characteristic class is defined as one-half that of the representation $D \otimes D$ in the square of the line bundle L . This is consistent with part (ii) of the following proposition, which establishes some basic properties of characteristic classes.

Proposition 2.1. (i) If D^* is the dual of D , then $\text{char}(D^*) = -\text{char } D$.
(ii) If D_1 and D_2 are representations of A on line bundles L_1 and L_2 over M , then

$$\text{char}(D_1 \otimes D_2) = \text{char}(D_1) + \text{char}(D_2) .$$

(iii) If (Φ, ϕ) is a morphism of Lie algebroids from $A \rightarrow M$ to $B \rightarrow N$, and D is a representation of B on a line bundle, then

$$\text{char}(\Phi^! D) = \tilde{\Phi}^*(\text{char } D) . \quad (2.1)$$

Proof. The proofs of (i) and (ii) are straightforward. Let us prove (iii). By Proposition 1.3, if D is a representation of B on the line bundle L , $\Phi^! D$ is a representation of A on $\phi^! L$. We denote by β the representative of $\text{char } D$ associated to ν , a nowhere-vanishing section of L , and by α the representative of $\text{char}(\Phi^! D)$ associated to $\phi^! \nu$. By definition, for all $a \in \Gamma A$,

$$\langle \alpha, a \rangle \phi^! \nu = a \cdot (\phi^! \nu) = \iota_a(d_{A, \phi^! L}(\phi^! \nu)) .$$

Since diagram (1.9) commutes,

$$d_{A, \phi^! L}(\phi^! \nu) = (\tilde{\Phi}^* \otimes \phi^!)(d_{B, L} \nu) .$$

Therefore, for $a \in \Gamma A$, $\alpha \in \Gamma(A^*)$, at any $m \in M$,

$$\begin{aligned} \langle \alpha_m, a_m \rangle (\phi^! \nu)_m &= \iota_{a_m}((\tilde{\Phi}^* \otimes \phi^!)(d_{B, L} \nu))(m) = \iota_{\Phi(a_m)}(d_{B, L} \nu)(\phi(m)) \\ &= \langle \beta_{\phi(m)}, \Phi(a_m) \rangle \nu_{\phi(m)} = \langle (\tilde{\Phi}^* \beta)_m, a_m \rangle (\phi^! \nu)_m . \end{aligned}$$

We have proved that $\alpha = \tilde{\Phi}^* \beta$, and (2.1) follows. \square

Each Lie algebroid $A \rightarrow M$ has a canonical representation D^A in the line bundle

$$L^A = \wedge^{\text{top}} A \otimes \wedge^{\text{top}} (T^*M) , \quad (2.2)$$

defined by

$$D_a^A(\omega \otimes \mu) = [a, \omega]_A \otimes \mu + \omega \otimes \mathcal{L}_{\rho_A a} \mu , \quad (2.3)$$

for all $a \in \Gamma A$, where $\omega \otimes \mu$ is a nowhere-vanishing section of L^A . Here $[\ , \]_A$ is the Gerstenhaber bracket on $\Gamma(\wedge^\bullet A)$, while \mathcal{L} denotes the Lie derivation. A section α of A^* satisfying

$$\langle \alpha, a \rangle \omega \otimes \mu = D_a^A(\omega \otimes \mu) , \quad (2.4)$$

for all $a \in \Gamma A$, is called the *modular cocycle for A* associated to the nowhere-vanishing section $\omega \otimes \mu$ of L^A , and the characteristic class it defines is called the *modular class of A* .

2.2 Definition of the modular class of a morphism

Using the cohomology pull-back operation associated to any morphism of Lie algebroids, we may make the following definition.

Definition 2.2. *Let $\text{Mod } A$ and $\text{Mod } B$ be the modular classes of Lie algebroids $A \rightarrow M$ and $B \rightarrow N$, and let (Φ, ϕ) be a morphism of Lie algebroids from $A \rightarrow M$ to $B \rightarrow N$. The relative modular class or simply the modular class of (Φ, ϕ) is the class, $\text{Mod } \Phi$, in the 1-cohomology of A defined by*

$$\text{Mod } \Phi = \text{Mod } A - \tilde{\Phi}^*(\text{Mod } B) .$$

This definition extends to general morphisms that of the modular class for base-preserving morphisms given in [12] [17].

It follows from the definition that, for Lie algebroids $A \rightarrow M$, $B \rightarrow N$ and $C \rightarrow R$, and morphisms (Φ, ϕ) from $A \rightarrow M$ to $B \rightarrow N$ and (Ψ, ψ) from $B \rightarrow N$ to $C \rightarrow R$,

$$\text{Mod}(\Psi \circ \Phi) = \text{Mod } \Phi + \tilde{\Phi}^*(\text{Mod } \Psi) . \quad (2.5)$$

In the particular case of a base-preserving morphism, $\tilde{\Phi}^*$ reduces to Φ^* . Therefore (2.5) (see also (6.1)) is the generalization of relation (3) of [17] to the Lie algebroid morphisms which are not necessarily base-preserving.

In the following sections, we shall prove that these relative modular classes are characteristic classes of representations, and we shall study various examples of such classes.

2.3 Modular classes of morphisms as characteristic classes

We will now show that the modular class of a Lie algebroid morphism from $A \rightarrow M$ to $B \rightarrow N$ is the characteristic class of a representation of A in a line bundle, thus generalizing Theorem 3.3 of [17] to the case of morphisms which are not necessarily base-preserving.

For any Lie algebroid morphism (Φ, ϕ) from $A \rightarrow M$ to $B \rightarrow N$, we set

$$L^\phi = L^A \otimes \phi^!((L^B)^*),$$

and

$$D^\Phi = D^A \otimes \Phi^!((D^B)^*),$$

where L^A and D^A are defined by (2.2) and (2.3), and $\Phi^!((D^B)^*)$ is defined in Proposition 1.3.

Theorem 2.3. *Let (Φ, ϕ) be a Lie algebroid morphism from $A \rightarrow M$ to $B \rightarrow N$.*

- (i) D^Φ is a representation of A on L^ϕ .
- (ii) The modular class, $\text{Mod } \Phi$, of the morphism (Φ, ϕ) is the characteristic class of the representation D^Φ .

Proof. The fact that D^Φ is a representation follows from the basic properties stated in Sections 1.2 and 1.3, while (ii) follows from parts (i) and (iii) of Proposition 2.1. \square

The following simple examples will be useful in the next sections.

Example 2.4. Let $A \rightarrow M$ be a Lie algebroid, $U \subset M$ an open subset and $A|_U \rightarrow U$ the restriction of $A \rightarrow M$ to U . Then the modular class of the inclusion morphism of $A|_U \rightarrow U$ into $A \rightarrow M$ vanishes.

Example 2.5. The modular class of a Lie algebroid isomorphism vanishes. In fact, a Lie algebroid isomorphism (Φ, ϕ) from $A \rightarrow M$ to $B \rightarrow N$ induces an isomorphism from the representation of $A \rightarrow M$ on the line bundle L^A to the representation of $B \rightarrow N$ on the line bundle L^B covering (Φ, ϕ) .

3 Pull-back Lie algebroids and pull-back morphisms

3.1 Definition of pull-backs

As in Section 1.3, we denote by $\phi^!B \rightarrow M$ the pull-back of a vector bundle $B \rightarrow N$ by a map $\phi : M \rightarrow N$, and for any section $b \in \Gamma B$, we denote by $\phi^!b$ the pulled-back section of $\phi^!B \rightarrow M$.

Definition 3.1. Let $B \rightarrow N$ be a Lie algebroid with anchor ρ_B . A map $\phi : M \rightarrow N$ is called *admissible* if the pull-back $B \oplus_{TN} TM$, whose fiber at $m \in M$ is $\{(b, u) \in B_{\phi(m)} \oplus T_m M \mid \rho_B b = (T\phi)u\}$, has a rank independent of m , in which case it is a vector sub-bundle of $\phi^!B \oplus TM$.

For instance, any surjective submersion is admissible. If $B \rightarrow N$ is transitive, *i.e.*, if ρ_B is surjective, any map into N is admissible. The inclusion $\mathcal{O} \rightarrow N$, an injective immersion, of any orbit \mathcal{O} of $B \rightarrow N$, *i.e.*, integral manifold of the distribution on M defined by the image of ρ_B , is admissible.

The following definition is due to Higgins and Mackenzie [13] [22].

Definition 3.2. *The pull-back of the Lie algebroid $(B \rightarrow N, \rho_B, [\cdot, \cdot]_B)$ by an admissible map $\phi : M \rightarrow N$ is the Lie algebroid $(\phi^!B \rightarrow M, \rho_B^!, [\cdot, \cdot]_{\phi^!B})$, where*

1. *the total space of $\phi^!B \rightarrow M$ is $B \oplus_{TN} TM$,*
2. *the anchor map $\rho_B^!$ is the projection onto the second component, and*
3. *the bracket is defined as follows. For $u, v \in \Gamma(TM)$, $b_i, c_i \in \Gamma B$, $f_i, g_i \in C^\infty(M)$, satisfying $\sum_i (f_i \circ \phi)(m) \rho_B(b_i(\phi(m))) = (T_m \phi)(u(m))$ and $\sum_i (g_i \circ \phi)(m) \rho_B(c_i(\phi(m))) = (T_m \phi)(v(m))$, for all $m \in M$,*

$$[(\sum_i f_i \otimes b_i, u), (\sum_i g_i \otimes c_i, v)]_{\phi^!B}$$

$$= \left(\sum_{i,j} f_i g_j \otimes [b_i, c_j]_B + \sum_i (u \cdot g_i) \otimes c_i - \sum_i (v \cdot f_i) \otimes b_i, [u, v]_{TM} \right).$$

For example, the pull-back of $B \rightarrow N$ by the inclusion $\mathcal{O} \rightarrow N$ of an orbit is a transitive Lie algebroid over \mathcal{O} (see [22]).

Remark 3.3. In the case when ϕ is a submersion, the bracket, $[\cdot, \cdot]_{\phi^!B}$, is the unique Lie algebroid bracket such that for any $b, c \in \Gamma B$ and $u, v \in \Gamma(TM)$ satisfying $\rho_B(b) = (T\phi)(u)$ and $\rho_B(c) = (T\phi)(v)$,

$$[(\phi^!b, u), (\phi^!c, v)]_{\phi^!B} = (\phi^![b, c]_B, [u, v]_{TM}). \quad (3.1)$$

The projection onto the first component is a morphism of Lie algebroids from $\phi^!B \rightarrow M$ to $B \rightarrow N$, which we denote by $\phi_B^!$.

As an example, we shall determine the pull-back Lie algebroid of the Lie algebroid $\mathcal{D}F$ of derivations on a vector bundle F , which exists since $\mathcal{D}F$ is transitive.

Proposition 3.4. *For any vector bundle F over N , and map $\phi : M \rightarrow N$, the Lie algebroids $\phi^!(\mathcal{D}F)$ and $\mathcal{D}(\phi^!F)$ are isomorphic.*

Proof. Recall that a representation of a Lie algebroid $A \rightarrow M$ on a vector bundle $E \rightarrow M$ can be seen as a Lie algebroid morphism from $A \rightarrow M$ to $\mathcal{D}E \rightarrow M$. Since there is a natural representation D of $\mathcal{D}F$ on F , it follows from Proposition 1.3 that $\Phi^!(D)$, where $\Phi = \phi_{\mathcal{D}F}^!$, is a representation of $\phi^!(\mathcal{D}F)$ on $\phi^!F$. Hence, there is a natural Lie algebroid morphism from $\phi^!(\mathcal{D}F)$ to $\mathcal{D}(\phi^!F)$. It suffices to prove that this vector bundle morphism is an isomorphism; this is an immediate consequence of the fact that the vector bundle $\phi^!(\mathcal{D}F)$ is locally isomorphic to the vector bundle $((F \otimes F^*) \oplus TN) \oplus_{TN} TM$, which is equal to $(F \otimes F^*) \oplus TM$, which in turn is locally isomorphic to $\mathcal{D}(\phi^!F)$. \square

A Lie algebroid morphism (Φ, ϕ) from $A \rightarrow M$ to $B \rightarrow N$ is said to be a *pull-back morphism* if $\phi : M \rightarrow N$ is admissible and if the Lie algebroids $A \rightarrow M$ and $\phi^! B \rightarrow M$ are isomorphic. In other words, (Φ, ϕ) is a pull-back morphism when Φ can be written as

$$\Phi = \phi_B^! \circ \Psi ,$$

where Ψ is a Lie algebroid isomorphism from $A \rightarrow M$ to $\phi^! B \rightarrow M$.

3.2 Pull-backs by transverse maps have vanishing modular classes

3.2.1 Transverse maps

We first recall some well-known facts about short exact sequences. Let $0 \rightarrow V_1 \xrightarrow{i} V_2 \xrightarrow{p} V_3 \rightarrow 0$ be a short exact sequence of finite-dimensional vector spaces, and let $r = \dim(V_3)$. There is a canonical isomorphism,

$$\wedge^{\text{top}} V_1 \otimes \wedge^{\text{top}} V_3 \xrightarrow{\cong} \wedge^{\text{top}} V_2 ,$$

defined by

$$X \otimes Y \mapsto i(X) \wedge \tilde{Y} ,$$

for all $X \in \wedge^{\text{top}} V_1$, $Y \in \wedge^{\text{top}} V_3$, where \tilde{Y} is any element in $\wedge^r V_2$ such that $(\wedge^r p) \tilde{Y} = Y$. Given an exact sequence $0 \rightarrow E_1 \xrightarrow{i} E_2 \xrightarrow{p} E_3 \rightarrow 0$ of vector bundles over a manifold M , the previous isomorphism applied pointwise yields a canonical isomorphism of line bundles,

$$\wedge^{\text{top}} E_1 \otimes \wedge^{\text{top}} E_3 \xrightarrow{\cong} \wedge^{\text{top}} E_2 ,$$

that we call the *canonical line-bundle isomorphism*.

We now define transverse maps.

Definition 3.5. A map $\phi : M \rightarrow N$ is said to be transverse to a Lie algebroid $B \rightarrow N$ if, for all $m \in M$,

$$(T\phi)(T_m M) + \rho_B(B_{\phi(m)}) = T_{\phi(m)} N.$$

Requiring ϕ to be a transverse map amounts to requiring that the following be an exact sequence of vector bundles,

$$0 \rightarrow B \oplus_{TN} TM \xrightarrow{i} \phi^! B \oplus TM \xrightarrow{p} \phi^! TN \rightarrow 0 , \quad (3.2)$$

where i is the inclusion map, and $p(\phi^! b + u) = \phi^!((T\phi)u - \rho_B b)$, for all $b \in B$ and $u \in TM$. In particular, the rank of $B \oplus_{TN} TM$ is constant, and the pull-back Lie algebroid $\phi^! B$ is well-defined. We have thus proved the following result.

Lemma 3.6. *Any map $\phi : M \rightarrow N$ transverse to a Lie algebroid $B \rightarrow N$ is admissible.*

Let (Φ, ϕ) be a pull-back morphism from a Lie algebroid $A \rightarrow M$ to $B \rightarrow N$, with ϕ transverse to $B \rightarrow N$. By definition, the Lie algebroids $A \rightarrow M$ and $\phi^! B \rightarrow M$ are isomorphic. To the short exact sequence (3.2) is associated a canonical isomorphism of line bundles,

$$\wedge^{\text{top}}(\phi^! TN) \otimes \wedge^{\text{top}} A \xrightarrow{\cong} \wedge^{\text{top}}(\phi^! B \oplus TM) = \wedge^{\text{top}}(\phi^! B) \otimes \wedge^{\text{top}}(TM) .$$

In turn, this isomorphism induces a canonical isomorphism of line bundles,

$$\wedge^{\text{top}} A \otimes \wedge^{\text{top}}(T^* M) \xrightarrow{\cong} \wedge^{\text{top}}(\phi^! B) \otimes \wedge^{\text{top}}(\phi^!(T^* N)) . \quad (3.3)$$

The left-hand side of (3.3) is the line bundle L^A (see (2.2) in Section 2.1), while the right-hand side is the line bundle $\phi^! L^B$. In conclusion, in the case of a pull-back morphism from $A \rightarrow M$ to $B \rightarrow N$ over a transverse map, (3.3) is a canonical line bundle isomorphism from L^A to $\phi^! L^B$ that we denote by ℓ^ϕ . In the next sections we shall prove that

$$\ell^\phi : L^A \xrightarrow{\cong} \phi^! L^B$$

is an isomorphism of A -modules. The proof of the following lemma is straightforward.

Lemma 3.7. *Let (Φ, ϕ) be a pull-back Lie algebroid morphism from $A \rightarrow M$ to $B \rightarrow N$, with ϕ transverse to $B \rightarrow N$, and let (Ψ, ψ) be a pull-back Lie algebroid morphism from $B \rightarrow N$ to $C \rightarrow R$, with ψ transverse to $C \rightarrow R$. Then,*

- (i) $(\Psi, \psi) \circ (\Phi, \phi)$ is a pull-back Lie algebroid morphism from $A \rightarrow M$ to $C \rightarrow R$, with $\psi \circ \phi$ transverse to $C \rightarrow R$, and
- (ii) the following diagram of line bundle isomorphisms over M , where $\phi^! \ell^\psi : \phi^! L^B \xrightarrow{\cong} \phi^! \psi^! L^C$ is the isomorphism obtained by pulling back the isomorphism $\ell^\psi : L^B \xrightarrow{\cong} \psi^! L^C$, commutes.

$$\begin{array}{ccc} L^A & \xrightarrow{\ell^\phi} & \phi^! L^B \\ & \searrow \ell^{\psi \circ \phi} & \swarrow \phi^! \ell^\psi \\ & \phi^! \psi^! L^C & \end{array}$$

3.2.2 Pull-backs by submersions

Any submersion $\phi : M \rightarrow N$ is transverse to any Lie algebroid over N . We shall now prove a property of the pull-backs by submersions.

Proposition 3.8. (i) When ϕ is a submersion, the canonical isomorphism, $\ell^\phi : L^A \xrightarrow{\cong} \phi^! L^B$, is an isomorphism of A -modules.

(ii) The modular class of a pull-back morphism (Φ, ϕ) from a Lie algebroid $A \rightarrow M$ to a Lie algebroid $B \rightarrow N$ vanishes whenever $\phi : M \rightarrow N$ is a submersion.

Proof. By Example 2.5 and Equation (2.5), we can assume that $A = \phi^! B$, and $(\Phi, \phi) = (\phi_B^!, \phi)$, and by Theorem 2.3, it suffices to prove that the canonical isomorphism, $\ell^\phi : L^A \xrightarrow{\cong} \phi^! L^B$, is an isomorphism of representations of $A \rightarrow M$. It suffices, for that purpose, to prove that ℓ^ϕ is an isomorphism of representations of $A|_U \rightarrow U$, after restriction to some neighborhood U of an arbitrary point $m \in M$. This restriction allows us to assume that all the line bundles involved in the computations below are topologically trivial, i.e., admit global nowhere-vanishing sections.

Let μ and ν be volume forms defined on $U \subset M$ and $\phi(U) \subset N$, respectively, and let σ be a nowhere-vanishing section of $\wedge^{\text{top}} B$ over $\phi(U)$. Henceforth, it will be understood that all sections of vector bundles over M (resp., N) are defined only over U (resp., $\phi(U)$).

Let $F \rightarrow M$ be the vertical tangent bundle of the submersion $\phi : M \rightarrow N$. Then $0 \rightarrow F \rightarrow TM \rightarrow \phi^! TN \rightarrow 0$ is an exact sequence of vector bundles over M , and there is a canonical isomorphism,

$$K : \wedge^{\text{top}} F \otimes \wedge^{\text{top}} (T^* M) \xrightarrow{\cong} \wedge^{\text{top}} ((\phi^! TN)^*) \xrightarrow{\cong} \phi^! (\wedge^{\text{top}} (T^* N)) ,$$

such that $K(\tau \otimes \mu) = \iota_\tau \mu$, for $\tau \in \Gamma(\wedge^{\text{top}} F)$. We define τ to be the unique section of $\wedge^{\text{top}} F$ over U such that $\iota_\tau \mu = \phi^* \nu$.

Since ϕ is a submersion, the sequence $0 \rightarrow F \rightarrow A \rightarrow \phi^! B \rightarrow 0$ is exact, so there also exists a canonical isomorphism,

$$J : \wedge^{\text{top}} (\phi^! B) \otimes \wedge^{\text{top}} F \xrightarrow{\cong} \wedge^{\text{top}} A .$$

Therefore, one can define a nowhere-vanishing section ω of $\wedge^{\text{top}} A$ over $U \subset M$ by

$$\omega = J(\phi^! \sigma \otimes \tau) .$$

It is routine to verify that the canonical isomorphism ℓ^ϕ satisfies

$$\ell^\phi(\omega \otimes \mu) = \phi^!(\sigma \otimes \nu) . \quad (3.4)$$

The modular cocycle for B with respect to the nowhere-vanishing section $\sigma \otimes \nu$ of $L^B = \wedge^{\text{top}} B \otimes \wedge^{\text{top}} (T^* N)$ is the section $\beta \in \Gamma(B^*)$ such that, for any $b \in \Gamma B$,

$$\langle \beta, b \rangle \sigma \otimes \nu = [b, \sigma]_B \otimes \nu + \sigma \otimes \mathcal{L}_{\rho_B b} \nu . \quad (3.5)$$

The modular cocycle for $\phi^!B$ with respect to the nowhere-vanishing section $\omega \otimes \mu$ of $L\phi^!B = \wedge^{\text{top}}(\phi^!B) \otimes \wedge^{\text{top}}(T^*M)$ is the section $\gamma \in \Gamma((\phi^!B)^*)$ such that, for any $c \in \Gamma(\phi^!B)$,

$$\langle \gamma, c \rangle \omega \otimes \mu = [c, \omega]_{\phi^!B} \otimes \mu + \omega \otimes \mathcal{L}_u \mu, \quad (3.6)$$

where $c = \phi^!b + u$, with $b \in \Gamma B$ and $u \in \Gamma(TM)$ such that $\rho_B b = (T\phi)u$.

We shall now prove the relation

$$\gamma - \tilde{\Phi}^* \beta = 0. \quad (3.7)$$

Assuming, without loss of generality, that, locally, $\sigma = b_1 \wedge \dots \wedge b_p$, with (b_1, \dots, b_p) a local basis of sections of B , where p is the rank of B , then $\omega = J(\phi^! \sigma \otimes \tau) = (\phi^! b_1 + u_1) \wedge \dots \wedge (\phi^! b_p + u_p) \wedge \tau$, where $u_k \in \Gamma(TM)$, $k = 1, \dots, p$. Therefore, by (3.1),

$$\begin{aligned} [c, \omega]_{\phi^!B} &= [\phi^!b + u, (\phi^!b_1 + u_1) \wedge \dots \wedge (\phi^!b_p + u_p) \wedge \tau]_{\phi^!B} \\ &= \sum_{k=1}^p (-1)^{k+1} [\phi^!b + u, \phi^!b_k + u_k]_{\phi^!B} \wedge (\phi^!b_1 + u_1) \wedge \dots \wedge \widehat{(\phi^!b_k + u_k)} \wedge \dots \wedge (\phi^!b_p + u_p) \wedge \tau \\ &\quad + (\phi^!b_1 + u_1) \wedge \dots \wedge (\phi^!b_p + u_p) \wedge [\phi^!b + u, \tau]_{\phi^!B} \\ &= \sum_{k=1}^p (-1)^{k+1} (\phi^![b, b_k]_B + [u, u_k]_{TM}) \wedge (\phi^!b_1 + u_1) \wedge \dots \wedge \widehat{(\phi^!b_k + u_k)} \wedge \dots \wedge (\phi^!b_p + u_p) \wedge \tau \\ &\quad + (\phi^!b_1 + u_1) \wedge \dots \wedge (\phi^!b_p + u_p) \wedge [u, \tau]_{TM}. \end{aligned}$$

This formula implies that

$$[c, \omega]_{\phi^!B} = J \left(\phi^! \sigma \otimes [u, \tau]_{TM} + \phi^![b, \sigma]_B \otimes \tau \right).$$

Now

$$\ell^\phi(\omega \otimes \mu) = \ell^\phi(J(\phi^! \sigma \otimes \tau) \otimes \mu) = \phi^! \sigma \otimes \iota_\tau \mu = \phi^! \sigma \otimes \phi^* \nu = \phi^*(\sigma \otimes \nu).$$

It follows that

$$\begin{aligned} \langle \gamma, c \rangle \ell^\phi(\omega \otimes \mu) &= \langle \gamma, c \rangle \phi^*(\sigma \otimes \nu) \\ &= \ell^\phi \left(J(\phi^!([b, \sigma]_B \otimes \tau) \otimes \mu) + J(\phi^! \sigma \otimes [u, \tau]_{TM}) \otimes \mu + J(\phi^! \sigma \otimes \tau) \otimes \mathcal{L}_u \mu \right) \\ &= \phi^![b, \sigma]_B \otimes \iota_\tau \mu + \phi^! \sigma \otimes (\iota_{[u, \tau]_{TM}} \mu + \iota_\tau \mathcal{L}_u \mu). \end{aligned}$$

From $[\mathcal{L}_u, \iota_\tau] = \iota_{[u, \tau]}$ and $\iota_\tau \mu = \phi^* \nu$ we obtain

$$\iota_{[u, \tau]_{TM}} \mu + \iota_\tau \mathcal{L}_u \mu = \mathcal{L}_u \iota_\tau \mu = \mathcal{L}_u(\phi^* \nu) = \phi^*(\mathcal{L}_{(T\phi)u} \nu) = \phi^*(\mathcal{L}_{\rho_B b} \nu).$$

Equation (3.7) is therefore proved and implies that

$$c \cdot (\omega \otimes \mu) = \phi^!(b \cdot (\sigma \otimes \nu)),$$

for any section $c \in \Gamma A$ of the form $c = \phi^!b + u$, with $\rho_B b = (T\phi)(u)$. This relation, together with Equations (3.4), (3.5) and (3.6), proves that ℓ^ϕ is an isomorphism of representations of $A \rightarrow M$. \square

3.2.3 Pull-backs by transverse maps

We shall need the following lemma.

Lemma 3.9. *Let $n \in N$ be a point in the base manifold of a Lie algebroid $B \rightarrow N$, q the rank of the anchor map ρ_B at the point n , and $q + s$ the dimension of N . There exist*

- *a surjective submersion ψ from a neighborhood V of n to an open subset W of \mathbb{R}^s , and*
- *a Lie algebroid $C \rightarrow W$*

such that

- *the anchor ρ_C vanishes at the point $\psi(n)$, and*
- *the restriction to V of the Lie algebroid $B \rightarrow N$ is isomorphic to the pull-back $\psi^!C$ of the Lie algebroid $C \rightarrow W$ by ψ .*

Proof. By a theorem of Dufour (see Theorem 8.5.1 in [6]), there exist local coordinates $(x_1, \dots, x_q, y_1, \dots, y_s)$ on N , centered at n , and a local trivialization $\alpha_1, \dots, \alpha_q, \beta_1, \dots, \beta_r$ of B , where $q + r = \text{rank } B$, such that, for $1 \leq i, j \leq q$ and $1 \leq a, b \leq r$,

$$\begin{cases} [\alpha_i, \alpha_j] &= 0 \\ \rho_B(\alpha_i) &= \frac{\partial}{\partial x_i} \\ [\beta_a, \alpha_j] &= 0 \\ \rho_B(\beta_a) &= \sum_{k=1}^s g_a^k \frac{\partial}{\partial y_k} \\ [\beta_a, \beta_b] &= \sum_{c=1}^r f_{ab}^c \beta_c, \end{cases}$$

where g_a^k and f_{ab}^c are functions of y_1, \dots, y_s , and $g_a^k(0, \dots, 0) = 0$. In particular, the last two relations define a Lie algebroid structure $C \rightarrow W$ in a neighborhood W of 0 in \mathbb{R}^s whose anchor vanishes at 0. All together, these formulas show that $B \rightarrow N$ is isomorphic to the pull-back of $C \rightarrow W$ by the projection $\psi(x_1, \dots, x_q, y_1, \dots, y_s) = (y_1, \dots, y_s)$. \square

We can now prove the main result of this section.

Theorem 3.10. *The modular class of a pull-back morphism (Φ, ϕ) from a Lie algebroid $A \rightarrow M$ to a Lie algebroid $B \rightarrow N$, when $\phi : M \rightarrow N$ is transverse to $B \rightarrow N$, vanishes.*

Proof. As in the proof of Proposition 3.8, it suffices to prove that ℓ^ϕ is an isomorphism of representations of $A|_U \rightarrow U$, after restriction to some neighborhood U of an arbitrary point $m \in M$.

By Lemma 3.9, there exists a neighborhood V of $n = \phi(m)$, a submersion $\psi : V \rightarrow W$, and a Lie algebroid $C \rightarrow W$ whose anchor vanishes at

$\psi(n) = \psi \circ \phi(m)$, such that $B|_V = \psi^! C$. Since the anchor map of $C \rightarrow W$ vanishes at the point $\psi(n)$, the differential at the point m of the transverse map $\psi \circ \phi$ is onto, so that, shrinking U , V and W if necessary, we can assume that $\psi \circ \phi$ is a submersion from U onto W .

By Proposition 3.8, the restriction of $\ell^{\psi \circ \phi}$ to U is an isomorphism of representations of $A|_U \rightarrow U$ from L^A to $(\psi \circ \phi)^! L^C$. Since ψ is a submersion, the restriction of ℓ^ψ to V also is an isomorphism of representations of $B|_V \rightarrow V$ from L^B to $\psi^! L^C$. Hence, the restriction of $\phi^! \ell^\psi$ to U is an isomorphism of representations of $A|_U \rightarrow U$ from $\phi^! L^B$ to $\phi^! \psi^! L^C$. By Lemma 3.7(ii), the restriction of ℓ^ϕ to U is also an isomorphism of representations of $A|_U \rightarrow U$ from L^A to $\phi^! L^B$, which completes the proof. \square

Proposition 3.11. *If (Φ, ϕ) is a Lie algebroid morphism from $A \rightarrow M$ to $B \rightarrow N$, and if ϕ is admissible with respect to $B \rightarrow N$, the map $\Phi: A \rightarrow B$ factors as a composition of maps $A \xrightarrow{\Phi'} \phi^! B \xrightarrow{\phi_B^!} B$, where Φ' is a morphism of Lie algebroids over M .*

Proof. When ϕ is admissible with respect to $B \rightarrow N$, the subbundle $\phi^! B$ of $\phi^! B \oplus TM$ is well-defined. For $m \in M$ and $a \in A_m$, the pair $(\phi^!(\Phi(a)), \rho_A(a))$ is in the fiber of $\phi^! B$ at m since $\rho_B \circ \Phi = T\phi \circ \rho_A$. Setting $\Phi'(a) = (\phi^!(\Phi(a)), \rho_A(a))$ defines a vector bundle map over the identity of M from A to $\phi^! B$ such that $\Phi = \phi_B^! \circ \Phi'$. To prove that Φ' is a Lie algebroid morphism, we write, for any $a \in \Gamma A$, $\phi^!(\Phi a) = \sum_i f_i \phi^!(b_i)$, where $f_i \in C^\infty(M)$ and $b_i \in \Gamma B$, and for a and $a' \in \Gamma A$, we express the bracket $[\phi^!(\Phi a), \phi^!(\Phi a')]_{\phi^! B}$ according to Definition 3.2. The result then follows from Theorem 4.6 of [3]. \square

The following is then a corollary of Theorem 3.10.

Proposition 3.12. *Let (Φ, ϕ) be a Lie algebroid morphism from $A \rightarrow M$ to $B \rightarrow N$ with the map $\phi: M \rightarrow N$ transverse to $B \rightarrow N$, and let $\Phi = \phi_B^! \circ \Phi'$ as above. Then*

$$\text{Mod } \Phi = \text{Mod } \Phi'.$$

This result shows that the modular class of any morphism which covers a transverse map is in fact equal to the modular class of a base-preserving morphism.

Remark 3.13. When ϕ is a transverse map, Proposition 1.3 has the following simple proof. The Lie algebroid morphism $D: B \rightarrow \mathcal{D}F$ over N gives rise to a Lie algebroid morphism $\phi^! D$ from $\phi^! B$ to $\phi^!(\mathcal{D}F)$ over M . Since, by Proposition 3.4, $\phi^!(\mathcal{D}F)$ is isomorphic to $\mathcal{D}(\phi^! F)$, the pull-back representation is obtained as the composition of Lie algebroid morphisms over M , $A \xrightarrow{\Phi'} \phi^! B \xrightarrow{\phi^! D} \mathcal{D}(\phi^! F)$, a representation of A on $\phi^! F$ satisfying condition (1.6).

3.3 A pull-back morphism with a non-vanishing modular class

We now give a counter-example which shows that Theorem 3.10 does not extend to the case in which ϕ is not transverse but only admissible in the sense of Definition 3.1.

Let $N = S^1 \times \mathbb{R}$ be a cylinder with coordinates (θ, x) , with $\theta \in S^1$ and $x \in \mathbb{R}$. Let $B \rightarrow N$ be the Lie subalgebroid of $TN \rightarrow N$ generated by the vector field

$$X = \frac{\partial}{\partial \theta} + x \frac{\partial}{\partial x}, \quad (3.8)$$

whose integral curves spiral in toward the invariant circle $x = 0$. The vector field X is a nowhere-vanishing section of $\Gamma(\wedge^{\text{top}} B) = \Gamma B$, while $dx \wedge d\theta$ is a nowhere-vanishing section of $\wedge^{\text{top}}(T^*N) = \wedge^2(T^*N)$. Let $\beta \in \Gamma(B^*)$ be the modular cocycle for B with respect to the nowhere-vanishing section $X \otimes (dx \wedge d\theta)$ of $\wedge^{\text{top}} B \otimes \wedge^{\text{top}}(T^*N)$. We shall compute β by evaluating it on X ,

$$\begin{aligned} \langle \beta, X \rangle X \otimes (dx \wedge d\theta) &= D_X^B(X \otimes (dx \wedge d\theta)) \\ &= X \otimes \mathcal{L}_X(dx \wedge d\theta) = X \otimes (dx \wedge d\theta). \end{aligned}$$

In conclusion, the modular class of $B \rightarrow N$ is the class of the section β of B^* which satisfies

$$\langle \beta, X \rangle = 1. \quad (3.9)$$

Let $M = S^1$ and let $\phi : M \rightarrow N$ be the inclusion map defined by $\phi(\theta) = (\theta, 0)$. We observe that, by construction, for all $m \in M$,

$$(T\phi)\left(\frac{\partial}{\partial \theta}\Big|_m\right) = X|_{\phi(m)}. \quad (3.10)$$

The pair $(T\phi, \phi)$ is a Lie algebroid morphism from $TM \rightarrow M$ to $TN \rightarrow N$ which takes values in $B \rightarrow N$ and yields a Lie algebroid morphism from $TM \rightarrow M$ to $B \rightarrow N$, which we denote also by $(T\phi, \phi)$. Since the Lie algebroid $TM \rightarrow M$ is unimodular, the modular class $\text{Mod}(T\phi)$ is $-(T^*\phi)(\text{Mod } B)$, and the latter is, according to (3.9) and (3.10), the class of the 1-form $-\beta$ on S^1 , where $\langle \beta, \frac{\partial}{\partial \theta} \rangle = 1$. In other words, the relative modular class of $(T\phi, \phi)$ is the class of $-d\theta$, and is therefore non-trivial.

Although the map ϕ is not transverse, it is admissible, being the inclusion of an orbit. It is easy to show that the pull-back Lie algebroid $\phi^!B \rightarrow M$ is the Lie algebroid $TM \rightarrow M$, and that the map $\phi^! : TM \rightarrow TN$ is $(T\phi, \phi)$. Therefore, we have found a Lie algebroid B over N and an admissible map $\phi : M \rightarrow N$ for which the modular class of the Lie algebroid morphism $\phi_B^! : \phi^!B \rightarrow B$ does not vanish.

3.4 The 1-cohomology of the pull-back by a surjective submersion

We first prove a lemma whose result will be used in the proof of Proposition 3.15.

Lemma 3.14. *Let $\phi : P \rightarrow M$ be a surjective submersion with 1-connected fibers, and let F be the vertical tangent bundle of ϕ . For any section $\beta \in \Gamma(F^*)$ whose restriction to each fiber is a closed 1-form, there exists a smooth function $f \in C^\infty(P)$ such that, for all $u \in F$,*

$$df(u) = \beta(u) .$$

Proof. Let $(V_i)_{i \in I}$ be an open cover of M which admits local sections of ϕ , $\sigma_i : V_i \rightarrow P$, and let $(\chi_i)_{i \in I}$ be a partition of unity associated to (V_i) . On $\phi^{-1}(V_i)$, one can define a smooth function f_i by $f_i(p) = \int_0^1 \beta(\gamma'(t)) dt$, where $\gamma(t)$ is any path of class C^1 contained in the fiber of p with endpoints $\gamma(0) = \sigma_i(\phi(p))$ and $\gamma(1) = p$. Since the fibers are 1-connected, this function is well-defined. It is differentiable and $df_i(u) = \beta(u)$, for any $u \in F|_{V_i}$. Therefore, the function $f = \sum_{i \in I} (\chi_i \circ \phi) f_i$ is a smooth function on P such that $df(u) = \beta(u)$, for all $u \in F$. \square

The following proposition (see [4] [8] [11]) describes an important property of the cohomology of pull-back Lie algebroids.

Proposition 3.15. *Let $A \rightarrow M$ be a Lie algebroid, and let $\phi : P \rightarrow M$ be a surjective submersion with 1-connected fibers. The map $(\widehat{\phi_A^{\text{II}}})^*$ induces an isomorphism from $H^1(A)$ to $H^1(\phi_A^{\text{II}})$.*

Proof. We denote $(\widehat{\phi_A^{\text{II}}})^*$ by $\widehat{\phi}$. We first prove that the map induced by $\widehat{\phi}$ is injective, making no assumption on the fibers of ϕ . Let $\alpha \in \Gamma(A^*)$ be a d_A -closed section such that $\widehat{\phi}\alpha$ is exact, i.e., such that $\widehat{\phi}\alpha = d_{\phi_A^{\text{II}}}f$, for some function $f \in C^\infty(P)$. We recall that the vertical tangent bundle F of the surjective submersion $\phi : P \rightarrow M$ is a subbundle of $\phi_A^{\text{II}} \rightarrow P$, and that $(\widehat{\phi}\alpha)(u) = 0$, for all $u \in F$. As a consequence, $df(u) = 0$, for all $u \in F$, and the function f is basic, i.e., there exists a function $g \in C^\infty(M)$ such that $f = g \circ \phi$. Since $(\phi_A^{\text{II}}, \phi)$ is a Lie algebroid morphism,

$$\widehat{\phi}\alpha = d_{\phi_A^{\text{II}}}f = \widehat{\phi}(d_A g) .$$

Since the map ϕ_A^{II} is a surjective bundle map, the above equation implies that $\alpha = d_A g$, and therefore that α is exact. In conclusion, the map induced by $\widehat{\phi}$ in cohomology is injective.

We shall now assume that the fibers of ϕ are 1-connected in order to prove that the map induced by $\widehat{\phi}$ is surjective. Let $\gamma \in \Gamma((\phi_A^{\text{II}})^*)$ be a $d_{\phi_A^{\text{II}}}$ -closed section. Its restriction to F is a fiberwise closed 1-form that

we denote by β . By Lemma 3.14, there exists a function $f \in C^\infty(P)$ such that $\beta(u) = df(u)$ for all $u \in F$, *i.e.*, for all vectors tangent to the fibers of ϕ . This implies that $\gamma' = \gamma - d_{\phi^!A} f$ is a closed section of $\Gamma((\phi^!A)^*)$ whose restriction to F vanishes. We shall prove that γ' is the image under $\widehat{\phi}$ of a section $\alpha \in \Gamma(A^*)$. For $a \in \Gamma A$ and $u \in \Gamma(TP)$ such that $\rho_A a = (T\phi)u$, let us set $(\phi^!)*\alpha(a) = \gamma'(\phi^!a + u)$, and let us show that the 1-form α on A is well-defined. The function $\gamma'(\phi^!a + u)$ does not depend on the choice of u since the restriction of γ' to F vanishes. It is also basic since, spelling out the fact that γ' is a closed section, we obtain, for any section $v \in \Gamma F$,

$$0 = (d_{\phi^!A} \gamma')(\phi^!a + u, v) = u \cdot \gamma'(v) - v \cdot \gamma'(\phi^!a + u) - \gamma'([\phi^!a + u, v]_{\phi^!A}) .$$

Since v and $[\phi^!a + u, v]_{\phi^!A} = [u, v]$ are sections of F ,

$$v \cdot \gamma'(\phi^!a + u) = 0 .$$

By construction, α is a closed section of A^* . In conclusion, $\widehat{\phi}$ induces a surjective map in cohomology. \square

4 The case of a regular base-preserving morphism

4.1 The modular class of a totally intransitive Lie algebroid

Let $A \rightarrow M$ be a totally intransitive Lie algebroid, *i.e.*, such that $\rho_A = 0$. Then, for $a \in \Gamma A$ and $\omega \otimes \mu \in \Gamma(L^A)$, where $L^A = \wedge^{\text{top}} A \otimes \wedge^{\text{top}}(T^*M)$,

$$D_a^A(\omega \otimes \mu) = [a, \omega]_A \otimes \mu .$$

By definition, $D_a^A(\omega \otimes \mu) = \langle \alpha, a \rangle \omega \otimes \mu$, where $\alpha \in \Gamma(A^*)$ is a modular cocycle for A , whence $\langle \alpha, a \rangle \omega = [a, \omega]_A$. Since the anchor of A vanishes, at each point $m \in M$,

$$\langle \alpha_m, a_m \rangle \omega_m = [a_m, \omega_m] ,$$

where the bracket is the Gerstenhaber bracket on the exterior algebra of the fiber at m which is a Lie algebra A_m . Since ω_m is a volume form on A_m , $\alpha_m \in A_m^*$ is a modular cocycle for the Lie algebra A_m , considered as a Lie algebroid over a point. Therefore

Proposition 4.1. *At each point of a totally intransitive Lie algebroid, the value of a modular cocycle is a modular cocycle for the fiber at that point.*

In degree 1, the linear space of cocycles of the Lie algebroid is the cohomology space $H^1(A)$ of the Lie algebroid, and similarly for each fiber Lie algebra. Therefore

Corollary 4.2. *In a totally intransitive Lie algebroid, the value of the modular class at a point is the modular class of the fiber at that point.*

Whenever A is a Lie algebra bundle, *i.e.*, locally trivial as a family of Lie algebras, the derived bundle $[A, A]$ of A also is a Lie algebra bundle [21] [22], and $H^1(A)$ is the space of sections of the bundle $(A/[A, A])^*$ whose rank is the co-rank of $[A, A]$. However, when the Lie algebroid A is only totally intransitive, a Lie algebroid 1-cocycle on A is a smooth family of Lie algebra cocycles for the fibers, but a cocycle for a given fiber might not be extendable to a cocycle on a neighborhood in M , as shown by the example of a 1-parameter family of semi-simple algebras whose Lie bracket degenerates to the zero bracket when the parameter equals zero.

4.2 Unimodular extensions of transitive Lie algebroids

Let C, A, B be Lie algebroids over the base M with anchors ρ_C, ρ_A and ρ_B , and brackets $[\cdot, \cdot]_C, [\cdot, \cdot]_A$ and $[\cdot, \cdot]_B$, respectively, and let

$$0 \longrightarrow C \xrightarrow{i} A \xrightarrow{\Phi} B \longrightarrow 0$$

be a Lie algebroid extension [21] [22], *i.e.*, i and Φ are base-preserving Lie algebroid morphisms, and the sequence is exact. Then C is totally intransitive since $\rho_C = \rho_A \circ i = \rho_B \circ \Phi \circ i = 0$. We assume that B is transitive, which implies that C is a Lie algebra bundle (see [21] [22] for the case $B = TM$).

We shall also assume that the Lie algebra bundle C is unimodular, in which case we call the Lie algebroid extension *unimodular*. By Proposition 4.1, this is the case if and only if each Lie algebra $C_m, m \in M$, is unimodular.

Finally, for simplicity, we shall assume that the vector bundles under consideration are orientable, although the results are valid without this hypothesis.

The adjoint action defines (see [22]) a representation $D^{A,C}$ of the Lie algebroid A on C by

$$i(D_X^{A,C} k) = [X, i(k)]_A, \quad (4.1)$$

for $X \in \Gamma A$ and $k \in \Gamma C$, and this representation induces a representation $D^{A,K}$ of the Lie algebroid A on $K = \wedge^{\text{top}} C$ such that, for $\lambda \in \Gamma K$,

$$(\wedge^r i)(D_X^{A,K} \lambda) = [X, (\wedge^r i)\lambda]_A, \quad (4.2)$$

where r is the rank of C and $[\cdot, \cdot]_A$ denotes the Gerstenhaber bracket on $\Gamma(\wedge^\bullet A)$. The unimodularity of C implies that λ can be chosen so that, for all $k \in \Gamma C$, $[k, \lambda]_C = 0$. Thus

$$(\wedge^r i)(D_{i(k)}^{A,K} \lambda) = [i(k), (\wedge^r i)\lambda]_A = (\wedge^r i)[k, \lambda]_C = 0.$$

As a consequence, the representation $D^{A,K}$ factors through the projection $\Phi : A \rightarrow B$, and yields a representation $D^{B,K}$ of B on K such that

$$D_{\Phi(X)}^{B,K} \lambda = D_X^{A,K} \lambda , \quad (4.3)$$

for all $X \in \Gamma A$.

An immediate consequence of Proposition 2.1(iii) is

Lemma 4.3. *The characteristic classes of the representations $D^{B,K}$ and $D^{A,K}$ are related by*

$$\text{char } D^{A,K} = \Phi^*(\text{char } D^{B,K}) .$$

A computation of the characteristic class of $D^{A,K}$ yields

Lemma 4.4. *The characteristic class of the representation $D^{A,K}$ is the modular class of Φ .*

Proof. Since the sequence $0 \rightarrow C_m \xrightarrow{i} A_m \xrightarrow{\Phi} B_m \rightarrow 0$ is exact for all $m \in M$, there exists an isomorphism of vector bundles, $\check{J} : \wedge^{\text{top}} A \otimes \wedge^{\text{top}} B^* \rightarrow \wedge^{\text{top}} C$, such that

$$(\wedge^r i) \check{J}(\omega \otimes \mu) = \iota_{\Phi^* \mu} \omega ,$$

for $\omega \in \Gamma(\wedge^{\text{top}} A)$ and $\mu \in \Gamma(\wedge^{\text{top}} B^*)$.

Let D^Φ be the representation of A on $\wedge^{\text{top}} A \otimes \wedge^{\text{top}} B^*$ such that the characteristic class of D^Φ is the modular class $\text{Mod } \Phi$ of Φ . Recall from [17] that

$$D_X^\Phi(\omega \otimes \mu) = [X, \omega]_A \otimes \mu + \omega \otimes \mathcal{L}_{\Phi(X)}^B \mu ,$$

for $X \in \Gamma A$. The isomorphism \check{J} intertwines the representations D^Φ and $D^{A,K}$ of A . In fact, for any $X \in \Gamma A$,

$$\begin{aligned} (\wedge^r i) \check{J}(D_X^\Phi(\omega \otimes \mu)) &= \iota_{\Phi^* \mu} [X, \omega]_A + \iota_{\mathcal{L}_X^A(\Phi^* \mu)} \omega = [X, \iota_{\Phi^* \mu} \omega]_A \\ &= [X, (\wedge^r i) \check{J}(\omega \otimes \mu)]_A = (\wedge^r i) D_X^{A,K}(\check{J}(\omega \otimes \mu)) . \end{aligned}$$

The conclusion follows since i is injective. \square

The preceding lemmas imply the following theorem.

Theorem 4.5. *For a unimodular Lie algebroid extension,*

$$0 \longrightarrow C \xrightarrow{i} A \xrightarrow{\Phi} B \longrightarrow 0 ,$$

the modular class of Φ satisfies

$$\text{Mod } \Phi = \Phi^*(\text{char } D^{B,K}) , \quad (4.4)$$

where $K = \wedge^{\text{top}} C$, and $D^{B,K}$ is defined by (4.3).

Remark 4.6. It follows from the proofs of Lemmas 4.3 and 4.4 that the relation (4.4) holds at the cochain level. More precisely, if θ is the d_A -cocycle defined by $D_X^\Phi(\omega \otimes \mu) = \langle \theta, X \rangle \omega \otimes \mu$, for all $X \in \Gamma A$, and η is the d_B -cocycle defined by $D_Y^{B,K} \lambda = \langle \eta, Y \rangle \lambda$, for all $Y \in \Gamma B$, then $\theta = \Phi^* \eta$ whenever $\check{J}(\omega \otimes \mu) = \lambda$.

4.3 Morphisms of constant rank with unimodular kernel

Let us assume more generally that the Lie algebroid morphism $\Phi: A \rightarrow A'$ over the identity of M has constant rank, and that the kernel $C = \ker \Phi$ is a unimodular Lie algebra bundle. Let B be the image of A under Φ , and let Φ_B be the morphism from A onto B induced by Φ . To express $\text{Mod } \Phi_B$, we can apply Theorem 4.5 to the unimodular Lie algebroid extension,

$$0 \longrightarrow C \xrightarrow{i} A \xrightarrow{\Phi_B} B \longrightarrow 0 .$$

Therefore

$$\text{Mod } A - \Phi_B^*(\text{Mod } B) = \Phi_B^*(\text{char } D^{B,K}) .$$

On the other hand, there is a representation $D^{B, \wedge^{\text{top}}(A'/B)}$ of B on $\wedge^{\text{top}}(A'/B)$. If i_B is the inclusion of B into A' , then a computation shows that

$$i_B^*(\text{Mod } A') - \text{Mod } B = \text{char } D^{B, \wedge^{\text{top}}(A'/B)} .$$

Since $\Phi = i_B \circ \Phi_B$, and since $\text{Mod } \Phi = \text{Mod } A - \Phi^*(\text{Mod } A')$, we obtain

Theorem 4.7. *Let $\Phi: A \rightarrow A'$ be a Lie algebroid morphism over the identity of M whose image B has constant rank and whose kernel is a unimodular Lie algebra bundle C . Then the modular class of Φ satisfies*

$$\text{Mod } \Phi = \Phi_B^*(\text{char } D^{B,K} - \text{char } D^{B, \wedge^{\text{top}}(A'/B)}) , \quad (4.5)$$

where $K = \wedge^{\text{top}} C$, and $D^{B,K}$ is defined by (4.3).

4.4 Examples

4.4.1 Transitive Lie algebroids

We apply the results of Section 4.3 to the case where $B = TM$ and Φ is the anchor map ρ_A of a transitive Lie algebroid A . The modular class of Φ is then the modular class of the Lie algebroid A . As in Section 4.2, we assume that the isotropy bundle $C = \ker \rho_A$ is unimodular.

In this case, the representation $D^{B,K}$ of $B = TM$ is a natural flat connection ∇ on $K = \wedge^{\text{top}} C$. The characteristic class $\text{char } D^{B,K}$ is the class of the closed 1-form α on M such that, for all $X \in \Gamma(TM)$ and $\lambda \in \Gamma K$, $\nabla_X \lambda = \langle \alpha, X \rangle \lambda$. Therefore Theorem 4.5 implies that

$$\text{Mod } A = \rho_A^*(\text{char}(D^{TM, \wedge^{\text{top}}(\ker \rho_A)})) .$$

In particular, if M is simply-connected, any transitive Lie algebroid over M with unimodular isotropy bundle is unimodular.

4.4.2 Regular Poisson structures

Let E be a Lie algebroid with a Poisson or twisted Poisson structure [16] [18] defined by a bivector $\pi \in \Gamma(\wedge^2 E)$. Assume that the structure is regular, *i.e.*, the associated map, $\pi^\sharp : E^* \rightarrow E$, is of constant rank. Applying Theorem 4.5 to $A = E^*$, $B = \text{Im}(\pi^\sharp)$ and $\Phi = \pi_B^\sharp$, the submersion from E^* to $B = \text{Im}(\pi^\sharp)$ defined by π^\sharp , we recover Lemma 2.3 of [18], which is written, in the notations that we are using here,

$$\text{Mod } \pi_B^\sharp = (\pi_B^\sharp)^*(\text{char } D^{B,K}) .$$

In this case, the representation $D^{B, \wedge^{\text{top}}(A'/B)}$ is dual to $D^{B,K}$, and therefore Theorem 4.7 yields

$$\text{Mod } \pi^\sharp = 2(\pi_B^\sharp)^*(\text{char } D^{B,K}) , \quad (4.6)$$

which is Theorem 2.5 of [18].

In the case of a Poisson manifold, (M, π) , the integrable distribution $B \subset TM$ defines the symplectic foliation, and (4.6) implies Corollary 9 of [4]. In particular (see [26]),

Proposition 4.8. *A regular Poisson manifold is unimodular if and only if its symplectic foliation admits an invariant transverse volume form.*

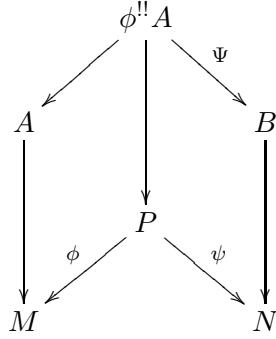
5 Generalized morphisms of Lie algebroids

5.1 The category of Lie algebroids with generalized Lie algebroid morphisms

There are several ways to define what could be called generalized morphisms of Lie algebroids, as an analogue to the definition of the generalized morphisms of Lie groupoids [2] [23]. Here, we follow the basic idea of Ginzburg [10], who defined Morita equivalences, but allowing enough generality to include all ordinary morphisms.

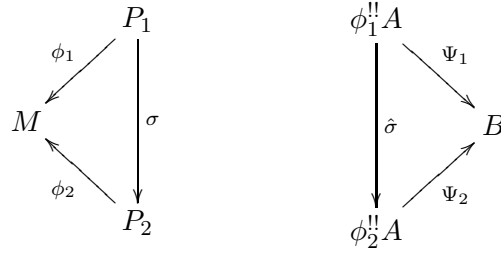
Definition 5.1. *A generalized morphism \mathcal{P} from a Lie algebroid $A \rightarrow M$ to a Lie algebroid $B \rightarrow N$ is a manifold P together with*

- *a surjective submersion ϕ from P to M , and*
- *a Lie algebroid morphism (Ψ, ψ) from $\phi^{!!}A \rightarrow P$ to $B \rightarrow N$.*



We denote a generalized morphism by a triple, $\mathcal{P} = (P, \phi, (\Psi, \psi))$.

An *isomorphism* between generalized morphisms $\mathcal{P}_i = (P_i, \phi_i, (\Psi_i, \psi_i))$, $i = 1, 2$, from $A \rightarrow M$ to $B \rightarrow N$ is a diffeomorphism $\sigma : P_1 \xrightarrow{\sim} P_2$ that intertwines both ϕ_1 and ϕ_2 , and Ψ_1 and Ψ_2 . More precisely, it is a diffeomorphism such that the following diagrams commute.



where $\hat{\sigma}$ is the Lie algebroid isomorphism from $\phi_1^!! A$ to $\phi_2^!! A$ induced by σ , defined by

$$\hat{\sigma}(\phi_1^! a + u) = \phi_2^! a + (T\sigma)u ,$$

for all $a \in A_{\phi_1(p)}$, $p \in P_1$, $u \in T_p P_1$, where $\rho_A a = (T\phi_1)u$. It is clear that the isomorphism of generalized morphisms is an equivalence relation.

We shall now prove that generalized morphisms can be composed. Let $\mathcal{P} = (P, \phi, (\Psi, \psi))$ be a generalized morphism from a Lie algebroid $A \rightarrow M$ to a Lie algebroid $B \rightarrow N$, and let $\mathcal{P}' = (P', \phi', (\Psi', \psi'))$ be a generalized morphism from a Lie algebroid $B \rightarrow N$ to a Lie algebroid $C \rightarrow R$. Consider the following data:

- the set $P'' = P \times_{\psi, N, \phi'} P'$, which is a manifold since ϕ' is a surjective submersion from P' to N ,
- the map ϕ'' defined by $\phi''(p, p') = \phi(p)$ for all $(p, p') \in P''$, which is a surjective submersion,
- the Lie algebroid $(\phi'')^!! A \rightarrow P''$ whose fiber over $(p, p') \in P''$ is

$$\left\{ (\phi'')^! a + u + v \in (\phi'')^! A_p \oplus T_p P \oplus T_{p'} P' \mid \rho_A a = (T\phi)u, (T\psi)u = (T\phi')v \right\}.$$

We can then define a vector bundle map Ψ'' from $(\phi'')^!A$ to C by

$$\Psi''((\phi'')^!a + u + v) = \Psi'((\phi')^!\Psi(\phi^!a + u) + v) .$$

This vector bundle map is, in fact, a Lie algebroid morphism covering the map $\psi'' : P \rightarrow R$ defined by $\psi''(p, p') = \psi'(p')$. Therefore $\mathcal{P}'' = (P'', \phi'', (\Psi'', \psi''))$ is a generalized morphism from $A \rightarrow M$ to $C \rightarrow R$. The compositions of isomorphic generalized morphisms are isomorphic, and the associativity of the composition is valid up to isomorphism. These properties justify the following definition.

Definition 5.2. (i) *The category of Lie algebroids with generalized Lie algebroid morphisms is the category \mathcal{Algd}' in which*

- *objects are Lie algebroids,*
- *arrows are isomorphism classes of generalized morphisms, and the identity arrow of a Lie algebroid $A \rightarrow M$ is the isomorphism class of the generalized morphism $(M, \text{id}_M, (\text{id}_A, \text{id}_M))$.*

(ii) *A Morita equivalence from a Lie algebroid $A \rightarrow M$ to a Lie algebroid $B \rightarrow N$ is a generalized morphism $\mathcal{P} = (P, \phi, (\Psi, \psi))$ where (Ψ, ψ) is a pull-back Lie algebroid morphism covering a surjective submersion $\psi : P \rightarrow N$.*

5.2 Modular classes of generalized morphisms

We now consider generalized Lie algebroid morphisms $\mathcal{P} = (P, \phi, (\Psi, \psi))$ such that the fibers of the map $\phi : P \rightarrow M$ are 1-connected. The composition of two such generalized morphisms satisfies the same condition (see [10]), and therefore generalized Lie algebroid morphisms with 1-connected fibers define a sub-category \mathcal{Algd}'_1 of the category \mathcal{Algd}' of Lie algebroids with generalized Lie algebroid morphisms.

We recall from Proposition 3.15 that a pull-back morphism covering a surjective submersion ϕ with 1-connected fibers induces an isomorphism of Lie algebroid cohomologies in degree 1, which we have denoted by $\hat{\phi}$. We can now define the modular class of a generalized morphism with 1-connected fibers.

Definition 5.3. *When $\mathcal{P} = (P, \phi, (\Psi, \psi))$ is a Lie algebroid generalized morphism with 1-connected fibers, its modular class is the image of the modular class of Ψ by $\hat{\phi}^{-1}$.*

The modular class of \mathcal{P} is an element of $H^1(A)$ which we denote by $\text{Mod } \mathcal{P}$. When $\mathcal{P} = (P, \phi, (\Psi, \psi))$ is a generalized morphism from $A \rightarrow M$ to $B \rightarrow N$ such that the fibers of ϕ are 1-connected, we denote by $\tilde{\mathcal{P}}^*$ the map from $H^1(B)$ to $H^1(A)$ induced by $\hat{\phi}^{-1} \circ \tilde{\Psi}^*$. It is clear from diagram (5.2) below that isomorphic generalized morphisms induce the same map on the 1-cohomology.

Theorem 5.4. (i) The modular classes of isomorphic generalized morphisms with 1-connected fibers are equal.

(ii) The modular class of the composition of two generalized morphisms \mathcal{P} and \mathcal{P}' with 1-connected fibers is

$$\text{Mod}(\mathcal{P}' \circ \mathcal{P}) = \text{Mod } \mathcal{P} + \tilde{\mathcal{P}}^*(\text{Mod } \mathcal{P}') . \quad (5.1)$$

(iii) The modular class of a Morita equivalence with 1-connected fibers vanishes.

Proof. (i) Let $\sigma : P_1 \rightarrow P_2$ be an isomorphism between generalized morphisms $\mathcal{P}_i = (P_i, \phi_i, (\Psi_i, \psi_i))$, $i = 1, 2$, from a Lie algebroid $A \rightarrow M$ to a Lie algebroid $B \rightarrow N$. Denote by $\hat{\sigma}$ the Lie algebroid isomorphism from $\phi_1^!! A$ to $\phi_2^!! A$ induced by σ . The following diagram commutes.

$$\begin{array}{ccc} & \phi_1^!! A & \\ (\phi_1)_A^!! \swarrow & \downarrow \hat{\sigma} & \searrow \Psi_1 \\ A & & B \\ (\phi_2)_A^!! \swarrow & \downarrow \hat{\sigma} & \searrow \Psi_2 \\ & \phi_2^!! A & \end{array} \quad (5.2)$$

Since $\hat{\sigma}$ induces an isomorphism of Lie algebroid cohomologies in degree 1, and since, by Proposition 3.15, the modular classes of $\phi_1^!! A$ and $\phi_2^!! A$ vanish, the result follows from the commutativity of the above diagram.

(ii) The projection onto the first component, $\chi : P'' = P \times_{\psi, N, \phi'} P' \rightarrow P$, is a surjective submersion, and there is a natural identification of Lie algebroids $\chi^!!(\phi''^!! A) \xrightarrow{\cong} (\phi'')^!! A$. There is also a Lie algebroid morphism $\Xi : (\phi'')^!! A \rightarrow (\phi')^!! B$, covering the projection onto the second component $\xi : P'' = P \times_{\psi, N, \phi'} P' \rightarrow P'$, defined by

$$\Xi((\phi'')^! a + u + v) = (\phi')^! \Psi(\phi^! a + u) + v ,$$

for $a \in A$, $u \in TP$, $v \in TP'$ such that $\rho_A a = (T\phi)u$ and $(T\psi)u = (T\phi')v$, and the following diagram of Lie algebroid morphisms commutes.

$$\begin{array}{ccccc} & & (\phi'')^!! A & & \\ & \swarrow (\phi'')_A^!! & \searrow \chi_{\phi''^!! A}^!! & \searrow \Xi & \searrow \Psi'' \\ & \phi^!! A & & & (\phi')^!! B \\ \swarrow \phi_A^!! & \searrow \Psi & & \swarrow (\phi')_B^!! & \searrow \Psi' \\ A & & B & & C \end{array}$$

By Proposition 3.15, all arrows pointing “south-west” induce isomorphisms of Lie algebroid cohomologies in degree 1. Since, by Proposition 3.8, each

has a vanishing modular class, the result follows from the commutativity of the above diagram.

(iii) is an immediate consequence of Proposition 3.8. \square

Theorem 5.4 (i) shows that the modular class of an isomorphism class of generalized morphisms with 1-connected fibers is well defined, while (5.1) generalizes formula (2.5).

6 Appendix: Representations of categories and the modular class

6.1 Representations of categories

Let \mathbf{R} be a commutative ring. A *representation* of a category \mathcal{C} over \mathbf{R} is a functor F from \mathcal{C} to the category of \mathbf{R} -modules [1] [9]. An *anti-representation* of a category \mathcal{C} is a representation of the opposite category, \mathcal{C}^{opp} .

When the category \mathcal{C} is a Lie groupoid $G \rightrightarrows X$, and \mathbf{R} is the field of real numbers, we assume that the vector spaces $F(x)$ associated to the objects, *i.e.*, points, x in X are the fibers of a smooth vector bundle. The assignment of vector bundle morphisms to the morphisms in G is just an action of G in the usual sense; we assume that this action is smooth.

Example 6.1. A representation of a Lie group considered as a groupoid over a point is a representation in the usual sense.

Example 6.2. A representation of the action groupoid associated to the action of a group H on a manifold X is the same as an H -equivariant vector bundle over X .

Remark 6.3. Representations of Lie algebroids are “infinitesimal representations”; they may occur as linearizations of representations of Lie groupoids.

6.2 H^1 as an anti-representation of the category of Lie algebroids

Assigning to each object A of $\mathcal{A}lgd$ the vector space $H^1(A)$ and to each Lie algebroid morphism $\Phi : A \rightarrow B$, where (A, B) is a pair of objects in $\mathcal{A}lgd$, the linear map $\tilde{\Phi}^* : H^1(B) \rightarrow H^1(A)$ defines an anti-representation of this category over \mathbb{R} , *i.e.*, a functor from $\mathcal{A}lgd^{\text{opp}}$ to the category of \mathbb{R} -modules, which we may denote by H^1 . The results of Section 5 show that H^1 is also an anti-representation of $\mathcal{A}lgd'_1$.

6.3 Cohomology of a category with values in an anti-representation.

The 0-cochains on a category \mathcal{C} with values in an anti-representation F assign to any object A of \mathcal{C} an element of $F(A)$. The k -cochains, $k \geq 1$, are maps from k -tuples of composable morphisms with source A , where A is an object of \mathcal{C} , to $F(A)$. The differential δ is defined by the usual formula. This definition is simpler than the one which we have found in the literature (see, *e.g.*, [1] [9]), but it defines the same object in the cases studied here, and seems therefore to be good enough for our limited purposes.

In particular, the coboundary of a 0-cochain u with values in F is the 1-cochain δu defined by $(\delta u)(\Phi) = u(A) - F(\Phi)(u(B))$, for any morphism $\Phi : A \rightarrow B$. If v is a 1-cochain with values in F , its coboundary δv assigns to a pair (Φ, Ψ) of composable morphisms, $A \xrightarrow{\Phi} B \xrightarrow{\Psi} C$, where (A, B, C) is a triple of objects in the category, the element $(\delta v)(\Phi, \Psi)$ which is the alternating sum $v(\Phi) - v(\Psi \circ \Phi) + F(\Phi)(v(\Psi))$ in $F(A)$.

6.4 The modular class of morphisms as a 1-coboundary.

A 0-cochain on $\mathcal{A}lgd$ with values in H^1 assigns to any object A of $\mathcal{A}lgd$ an element of $H^1(A)$. The modular class of Lie algebroids is a 0-cochain Mod on $\mathcal{A}lgd$ (resp., on $\mathcal{A}lgd'_1$) with values in the anti-representation H^1 .

A 1-cochain assigns to a morphism $\Phi : A \rightarrow B$ an element of $H^1(A)$. The coboundary of a 0-cochain u on $\mathcal{A}lgd$ with values in H^1 is the 1-cochain δu defined by $(\delta u)(\Phi) = u(A) - \tilde{\Phi}^*(u(B))$. In particular,

$$(\delta \text{Mod})(\Phi) = \text{Mod } A - \tilde{\Phi}^*(\text{Mod } B) .$$

The same argument holds for generalized morphisms. Thus, we have the following result.

Proposition 6.4. *The modular class of morphisms (resp., generalized morphisms with 1-connected fibers) of Lie algebroids is the coboundary of the modular class of Lie algebroids, viewed as a 0-cochain on $\mathcal{A}lgd$ (resp., on $\mathcal{A}lgd'_1$) with values in the anti-representation H^1 .*

If v is a 1-cochain, its coboundary δv assigns to a composable pair $A \xrightarrow{\Phi} B \xrightarrow{\Psi} C$ of Lie algebroid morphisms the element in $H^1(A)$,

$$(\delta v)(\Phi, \Psi) = v(\Phi) - v(\Psi \circ \Phi) + \tilde{\Phi}^*(v(\Psi)) .$$

As a consequence of the relation $\delta^2 = 0$, if $v = \delta u$,

$$v(\Psi \circ \Phi) = v(\Phi) + \tilde{\Phi}^*(v(\Psi)) . \tag{6.1}$$

The preceding relation applied to $v = \delta \text{Mod}$ yields Equations (2.5) and (5.1).

Remark 6.5. One may ask whether there are any characteristic classes of Lie algebroid morphisms which satisfy (6.1) but which do not arise from characteristic classes of the objects in the category of Lie algebroids. In fact, there are none, for the following general reason. In any category \mathcal{C} with an initial object $\{\text{pt}\}$, *i.e.*, the opposite of a category with a final object, such as the category of Lie algebroids, the cohomology in degree 1 is trivial. In fact, for each object A of \mathcal{C} , let us denote by p_A the morphism $A \rightarrow \{\text{pt}\}$, and let v be a 1-cochain on \mathcal{C} . Let u be the 0-cochain defined by $u(A) = v(p_A)$. If v is a 1-cocycle, for any morphism $\Phi : A \rightarrow B$, $v(p_B) = v(p_A \circ \Phi) = v(\Phi) + F(\Phi)(v(p_A))$, whence $v(\Phi) = u(B) - F(\Phi)(u(A))$, and v is the coboundary of u .

On the other hand, there do exist nontrivial cocycles on other interesting categories. We learned the following example from Peter Teichner. Let \mathcal{C} be the category whose objects are smooth manifolds and whose morphisms are immersions $f : M \rightarrow N$ equipped with an almost complex structure on the normal bundle. (Composition of morphisms is composition of immersions and direct sum of complex structures.) The de Rham 1-cohomology is a representation of this category, and the Chern class of the normal bundle is a 1-cocycle which is not a coboundary.

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